Indian J. pure appl. Math., **28**(7) : 969-1003, June 1997

ASYMPTOTIC METHODS IN MAGNETOCONVECTION

N. RUDRAIAH

INSA Senior Scientist, National Research Institute for Applied Mathematics, 492/G 7th Cross, 7th Block (West), Jayanagar, Bangalore 560 082

and

UGC-DSA In Fluid Mechanics, Department of Mathematics, Bangalore University, Bangalore 560 001

(*Received* 9 *June* 1995; *after revision* 27 *January* 1997; *accepted* 14 *April* 1997)

The effects of Lorentz force and non-uniform temperature gradient on the onset of magnetoconvection in an electrically conducting horizontal Boussinesq fluid layer permeated by a uniform transverse magnetic field are studied analytically using linear stability analysis by specifying constant temperature or constant heat flux at the boundaries. It is shown that when the Chandrasekhar number $Q \rightarrow \infty$ the correct asymptotic value of the critical Rayleigh number, R_c, can be obtained from the non-viscous MHD equations using a single-term Galerkin expansion. The criterion for the onset of magnetoconvection is determined using a regular perturbation technique with wave-number as perturbation parameter. The method of matched asymptotics is used to predict explicitly the effect of the Hartmann boundary layer (that exists at the rigid boundary for large values of Q) on the onset of magnetoconvection. It is shown that the effect of the Hartmann boundary layer is to increase the asymptotic value of R_c by an amount proportional to the value of the Hartmann number M. We find that the ratios R_{c_i}/R_{c_1}, where R_{c_i} (i = 1 to 6) are the asymptotic values of R_c for different nonlinear temperature profiles, are independent of Q but dependent on thermal depth ε. It is also shown that the power law for asymptotic values of R_{c_i} depends crucially on the nature of heating and not on the nature of the boundaries.

Key Words : Asymptotic Methods; Magnetoconvection

INTRODUCTION

Linear and nonlinear study of natural convection in an electrically conducting fluid in the presence of a magnetic field (known as magnetoconvection) caused either by buoyancy force[1-10] or by surface tension[11] or by both[12,13] has received considerable interest during the last three decades owing to its astrophysical, geophysical and engineering applications. So far the study of magnetoconvection, both theoretical and experimental, has been concerned with a layer of an electrically conducting fluid bounded by isothermal boundaries with unfirom heating from below and cooling from above. In the case of stress free boundaries, specifying a constant temperature is more restrictive than specifying constant heat flux[14]. The difficulty in the case of specifying constant heat flux at the boundaries is the non-existence of steady state basic temperature profile because it cannot be determined uniquely. In that case the basic temperature profile depends explicitly on position and time. This has to be determined by solving the coupled momentum and energy equations which makes the problem very complicated. Because of this difficulty much work on magnetoconvection has not been done by specifying a constant heat flux at the boundaries inspite of its applications in astrophysics, geophysics and in material science processing particularly in the manufacture of semiconductor devices and in the theory and modelling of crystal growth process. To overcome this difficulty, in the case of marginal state discussed in this paper, we consider a series of temperature profiles based on a simplification in the form of quasi-static approximation[10] that consists of freezing the temperature distribution at a given instant of time. This assumption may be justified using two different time scales; one the thermal time scale d^2/K and the other the viscous time scale d^2/ν, where K is the thermal diffusivity, ν is the kinematic viscosity and d is the depth of the fluid layer. The study of magnetoconvection specifying a constant heat flux at the boundaries is one of the objectives of the present paper by adopting a series of temperature profiles based on a simplification explained above. It is known that[1] depending on the values of Chandrasekhar number, Q, and Prandtl numbers, both steady and overstable motions are possible in the study of magnetoconvection. Because of the extra complexity of the calculations involved, we shall defer the study of the overstable state and consider in this paper only the marginal state. Even with these simplications, the solution of the stability problem poses a problem because the temperature gradient varies with depth. This problem is overcome by using a single term Galerkin expansion, which eliminates the use of elaborate numerical computation.The objectives of this article are the following :

(i) To find analytically the critical Rayleigh number, R_c, in the presence of magnetic field and a non-uniform temperature gradient by a method that eliminates using elaborate numerical computations[1] and obtain results close to those obtained by experiments. We show that Galerkin and perturbation techniques are well suited for this purpose.

(ii) To show that the asymptotic dependence of Q on R_c is based on the nature of heating (i.e., non-uniform temperature gradient) and the nature of the boundaries. For uniform temperature gradient with isothermal boundaries, Chandrasekhar[1] has proved that R_c obeys asymptotically the power law $R_c \to \pi^2 Q$ for large values of Q in the case of free-free isothermal boundaries. From the results of his calculations he[1] states that "It appears that the same power law (i.e., $R_c \to \pi^2 Q$) with the same constant of proportionality hold for rigid-rigid and rigid-free isothermal boundaries." In this paper, we show that Chandrasekhar's power law $R_c \to \pi^2 Q$ is true only for free-free isothermal boundaries and in other boundary combination although R_c obeys the same power law but differs by a constant of proportionality.

(iii) To show the limitations of the use of Galerkin technique in the case of rigid-rigid boundaries with constant heat flux, we show that the polynomial type of trial functions used in the Galerkin technique does not take care of the boundary layer effect that inevitably arises at the rigid boundaries for large values of Q. In that case, we propose an alternate analytical method based on a regular perturbation technique. This gives reasonable results with minimum mathematics.

(iv) To determine quantitatively the effect of boundary layer on R_c using a singular perturbation technique.

To achieve these objectives the basic equations and the basic state are discussed in section 2. The linear stability theory subject to infinitesimal disturbances with different boundary combinations is discussed in section 3 specifying constant temperature and in section 4 specifying constant heat flux. Asymptotic values of R_c for large values of Q are also discussed in these sections. Regular and singular perturbation techniques are respectively used in sections 5 and 6 to determine the effect of boundary layer on R_c. The final section is devoted to the results and conclusions.

2. Mathematical Formulation

Consider a Boussinesq electrically conducting fluid that fills a thin horizontal layer of finite depth d, extending to infinity in the two horizontal directions x and y with an imposed transverse uniform magnetic field H_0, in the vertical z-direction. For this system the basic equations are given in section 2.1.

2.1 Basic Equations

The basic equations of motion for a Boussinesq electrically conducting fluid are

$$\begin{bmatrix} D-\nu\nabla^2 & 0 & 0 & -\mu/\rho_0\overline{D} & 0 & 0 & D_1/\rho_0 & 0 & 0 \\ 0 & D-\nu\nabla^2 & 0 & 0 & -\mu/\rho_0\overline{D} & 0 & D_2/\rho_0 & 0 & 0 \\ 0 & 0 & D-\nu\nabla^2 & 0 & 0 & -\mu/\rho_0\overline{D} & D_3/\rho_0 & g/\rho_0 & 0 \\ -\overline{D} & 0 & 0 & D-\nu_m\nabla^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\overline{D} & 0 & 0 & D-\nu_m\nabla^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\overline{D} & 0 & 0 & D-\nu_m\nabla^2 & 0 & 0 & 0 \\ D_1 & D_2 & D_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_1 & D_2 & D_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D-K\nabla^2 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \\ H_x \\ H_y \\ H_z \\ p \\ \rho \\ T \end{bmatrix} = 0, \quad \ldots (2.1)$$

with the equation of state

$$\rho = \rho_0 [1 - \alpha (T - T_0)], \qquad \text{... (2.2)}$$

where

$$D_0 = \frac{\partial}{\partial t}, \quad D_1 = \frac{\partial}{\partial x}, \quad D_2 = \frac{\partial}{\partial y}, \quad D_3 = \frac{\partial}{\partial z},$$

$$\nabla^2 = \nabla_1^2 + D_3^2, \ \nabla_1^2 = D_1^2 + D_2^2, \ D = D_0 + uD_1 + vD_2 + wD_3,$$

$\overline{D} = H_x D_1 + H_y D_2 + H_z D_3$, (u, v, w) and (H_x, H_y, H_z) are x, y and z components of velocity and magnetic field respectively, p is the pressure, ρ is the density, T is the temperature, T_0 is the temperature at $\rho = \rho_0$, ν is the kinematic viscosity, $\nu_m (= 1/\mu\sigma)$ is the magnetic viscosity, μ is the magentic permeability, σ is the electrical conductivity, K is the thermal diffusivity, (x, y, z) are the Cartesian co-ordinates and t is the time.

These equations have to be solved using suitable boundary conditions which depend on the nature of the boundaries. These will be discussed later.

2.2 *Basic State*

In the basic state the fluid layer is isothermal and at rest and obeys the hydrostatic balance. At some time $t = 0$, heat is applied in an arbitrary manner with respect to time but uniformly with respect to the lower bounding surface. Let ΔT be the temperature difference between the lower and upper surfaces with lower boundary at a greater temperture than the upper boundary. In other words, the basic temperature T_b, finds its origin in transient heating or cooling at the boundaries so that it depends explicitly on time and the vertical co-ordinate z. This T_b has to be determined by solving

$$\frac{\partial T_b}{\partial t} = K \frac{\partial^2 T_b}{\partial z^2} . \qquad \text{... (2.3)}$$

In this paper we deal with the marginal state and hence we adopt a series of temperature profiles (see section 3.4) based on a simplification in the form of a quasi-static approximation that consists of freezing the temperature distribution at a given instant of time (Currie[15]). In this approximation, (2.3) may be written as

$$\frac{dT_b}{dz} = -\frac{\Delta T}{d} f(z), \qquad \text{... (2.4)}$$

where $f(z)$ represents a non-uniform temperature gradient and is such that

$$\int_0^1 f(z)\, dz = 1. \qquad \text{... (2.5)}$$

This type of basic temperature profile facilitates the use of constant heat flux at the boundaries.

In the remaining part of this paper we study the stability of this basic state subject to infinitesimal disturbances, known as linear stability.

3. Linear Stability Analysis

The determination of the condition for the onset of convection is the realm of linear theory. This is investigated in this section subject to infinitesimal disturbances. Suppose that the basic state is slightly disturbed. Then the linearised version of (2.1) after eliminating the pressure by operating curl twice on the momentum equation and following Chandrasekhar[1], we get

$$\frac{\partial}{\partial t}(\nabla^2 w) = \alpha g\ \nabla_1^2 T + \nu\ \nabla^4 w + \frac{\mu}{\rho_0} H_0 \frac{\partial}{\partial z}(\nabla^2 H_z), \qquad \text{... (3.1)}$$

$$\frac{\partial T}{\partial t} = \frac{\Delta T}{d} f(z) w + K\ \nabla^2 T, \qquad \text{... (3.2)}$$

and

$$\frac{\partial H_z}{\partial t} = H_0 \frac{\partial w}{\partial z} + \nu_m\ \nabla^2 H_z \qquad \text{... (3.3)}$$

Making these equations dimensionless using $d, d^2/\nu, \nu/d, \nu/ad \left(\frac{\Delta T \nu}{\alpha g d K}\right)^{1/2}$ and H_0 as the scales for length, time, velocity, temperature and magnetic field respectively, and seeking the solutions of the form

$$(W, T, H) = (W(z), T(z), H(z))\ \exp\ [i\ (lx + my) + p_0 t], \qquad \text{... (3.4)}$$

we get

$$\sigma(D^2 - a^2)\ W = -aR^{1/2}\ T + \frac{Q}{P_m} D(D^2 - a^2)\ H + (D^2 - a^2)^2\ W, \qquad \text{... (3.5)}$$

$$\sigma P_r\ T = aR^{1/2} f(z) W + (D^2 - a^2) T, \qquad \text{... (3.6)}$$

and

$$\sigma P_m H = P_m\ DW + (D^2 - a^2) H, \qquad \text{... (3.7)}$$

where $D = d/dz$, $a = (l^2 + m^2)^{1/2}$, l and m are the horizontal wave numbers, $R = \frac{\alpha g\ \Delta\ T d^3}{\nu K}$ is the Rayleigh number, $Q = \frac{\mu H_0^2\, d^2}{\rho_0\, \nu\, \nu_m}$ is the Chandrasekhar number which is the square of the Hartmann number M, $P_m = \nu/\nu_m$ is the magnetic Prandtl number, $P_r = \nu/k$ is the Prandtl number and $\sigma = \frac{p_0\, d^2}{\nu}$ is the time constant.

3.1 Boundary Conditions

These equations will be solved using the appropriate boundary conditions depending on the nature of the boundaries. We consider both rigid and free boundaries with isothermal or constant heat flux (i.e., adiabatic) conditions. On the rigid boundaries

$$W = DW = 0. \qquad \text{... (3.8)}$$

and on the stress-free boundaries

$$W = D^2 W = 0. \qquad \text{... (3.9)}$$

The boundary conditions on H will depend on whether the boundaries are electrically conducting or non-conducting. In the case of linear theory, the boundary conditions on H are linked with the conditions on w through the magnetic induction equation (3.7). On isothermal boundaries we specify $T = 0$ and for constant heat flux at the boundaries we specify $DT = 0$.

The temperature scale has been chosen so that R appears symmetrically in the energy and momentum equations rather than just in one equation. Though this choice of temperature scale is not essential for our purpose and has no physical implications, it enables a variational principle to be established for the present set of equations. This form is useful to establish that the eigenvalues of the system are stationary.

3.2 Theorems Concerned with Eigenvalues

If $f(z) = 1$, (3.5)-(3.7) are those given in Chandrasekhar[1] with a slightly different normalisation. The presence of $f(z)$ ($\neq 1$) in (3.6) preclude the use of the method of Chandrasekhar[1]. In this paper, therefore, we use a more convenient and general Galerkin method that gives reasonable results for various temperature gradient models using simple, polynomial trial functions for the lowest eigenmodes. This seeks to obtain an approximate solution of the differential equation with the given boundary conditions and may not exactly satisfy the differential equations. This leads to a residual when the trial functions are substituted into the differential equation. The Galerkin method requires the residual to be orthogonal to each individual trial function. This requires that the principle of exchange of stability is valid and the eigenvalues of the problem to be real and stationary which are established through the following Theorems 1 to 3 (Finlayson[16]). Chandrasekhar[1] had proved the principle of exchange of stability when $f(z) = 1$. Here we extend this fo $f(z) \neq 1$.

Theorem 1 — *For $K < \nu_m$, over stability cannot occur and the principle of the exchange of stability is valid.*

PROOF : We establish this theorem by deriving a moment equation. Eliminating H between (3.5) and (3.7) and rearranging them, we get

$$(D^2 - a^2)\left[(D^2 - a^2 - \sigma)(D^2 - a^2 - \sigma P_m) - QD^2\right] w$$

$$= aR^{1/2}(D^2 - a^2 - \sigma P_m)\, T \qquad \text{... (3.8)}$$

$$(D^2 - a^2 - \sigma P_r)\, T = -\, aR^{1/2} f(z) w. \qquad \text{... (3.9)}$$

To derive the moment equations we multiply (3.8) by W^m and (3.9) by T^m and integrate over z from 0 to 1. Many terms can be integrated by parts and using the boundary conditions discussed in section 3.1, we get

$$\langle\, D^3 W^m D^3 W + a_0 D^2 W^m D^2 W + a_1 DW^m Dw + a_2 W^m W \,\rangle$$

$$= aR^{1/2} \langle\, DW^m Dt + (a^2 + \sigma P_m)\, W^m T \,\rangle + \Big[\, DW^m D^4 w - D^2 W^m D^3 w \,\Big]_0^1$$

$$\text{... (3.10)}$$

$$\langle\, DT^m DT + (a^2 + \sigma P_r)\, T^m T \,\rangle = aR^{1/2} \langle\, fWT^m \,\rangle\, 6 \qquad \text{... (3.11)}$$

where $a_0 = 3a^2 + \sigma(1 + P_m) + Q$, $a_1 = 3a^4 + 2a^2\sigma(1 + P_m) + \sigma^2 P_m + Qa^2$, $a_2 = a^6 + \sigma(1 + P_m)\, a^4 + a^2 \sigma^2 P_m$. Here $m = 0$, gives the moment equation and $m = 1$, gives the energy equation. Recently, Mikaellian[17] has shown that moment equation gives the best result and does not suffer from the ambiguities of satisfying some of the higher order boundary conditions. We, therefore, use moment equations to prove the Theorem 1.

For $m = 0$, we obtain

$$a_2 \langle\, w \,\rangle = aR^{1/2} (a^2 + \sigma P_m) \langle\, T \,\rangle \qquad \text{... (3.12)}$$

and

$$(a^2 + \sigma P_r) \langle\, T \,\rangle = aR^{1/2} \langle\, fw \,\rangle \qquad \text{... (3.13)}$$

Substituting $\langle\, T \rangle$ from (3.13) into (3.12) and substituting the expression a_2 and rearranging we get

$$a^6 + \sigma(1 + P_m + P_r)\, a^4 + a^2 \sigma^2 (P_m + P_r + P_m P_r)$$

$$+ \sigma^3 P_m P_r = a^2 RF + \sigma R P_m F_1 \qquad \text{... (3.14)}$$

where $$F = \frac{\langle fW \rangle}{\langle\, W \,\rangle}. \qquad \text{... (3.15)}$$

Since we are interested in oscillatory motions, following Chandrasekhar[1], we set $\sigma = i\sigma_i$, with σ_i real, in (3.14), and obtain

$$a^6 + i\sigma_i (1 + P_m + P_r)\, a^4 - a^2 \sigma_i^2 (P_m + P_r + P_m P_r)$$

$$+ i\sigma_i^3 P_m P_r = a^2 RF + i\sigma_i R P_m F \qquad \text{... (3.16)}$$

Since W in general is complex, F in (3.15) is complex for $f(z) \neq 1$. We are interested in finding eigenvalues and therefore look for solution of the form W = Constant $W_1(z)$ (as in[1]) where constant in general complex and $W_1(z)$ is a trial function chosen suitably to satisfy the boundary condition. In this case F is real.

Equating separately the real and imaginary parts of (3.16), we obtain

$$a^4 - (P_m + P_m P_r + P_r)\,\sigma_i^2 = RF \qquad \text{... (3.17)}$$

and

$$(1 + P_m + P_r)\,a^4 - \sigma_i^2 P_m P_r = P_m RF \qquad \text{... (3.18)}$$

substituting RF from (3.17) into (3.18) we obtain on simplification

$$P_m^2 (1 + P_r)\,\sigma_i^2 = (P_m - P_r)\,a^4 - (1 + P_m)\,a^4 \qquad \text{... (3.19)}$$

from (3.19) we conclude that solution describing overstability cannot occur even in the case of non-uniform temperature gradient (i.e., $f(z) \neq 1$) if

$$P_m < P_r \text{ i.e., } k < \nu_m \qquad \text{... (3.20)}$$

for σ_i^2 would then be negative, contrary to hypothesis. Hence the theorem.

***Theorem* 2** — *If the solution* $W(r, R, a, Pm, Q) = \begin{bmatrix} W \\ T \\ H \end{bmatrix}$ *of the system (3.5)-(3.7) for a steady marginal state and the corresponding eigenvalues are continuously differentiable functions of a,* P_m *and Q, the eigenvalues of the system (3.5)-(3.7) with* $\sigma = 0$*, are real.*

PROOF : We can write the system (3.5)-(3.7) using $\sigma = 0$ in the operator form

$$LW - \lambda L_0 W = 0, \qquad \text{... (3.21)}$$

where

$$L = \begin{bmatrix} (D^2 - a^2)^2 & 0 & \dfrac{Q}{P_m} D(D^2 - a^2) \\ 0 & -(D^2 - a^2) & 0 \\ P_m D & 0 & D^2 - a^2 \end{bmatrix},$$

$$L_0 = \begin{bmatrix} 0 & 1 & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} W \\ T \\ H \end{bmatrix}.$$

The adjoint set of (3.8) is

$$L^* W^* - \lambda^* L_0^* W^* = 0. \qquad \text{... (3.22)}$$

We consider two eigenfunctions W and W^*. Multiplying (3.21) by W and (3.22) by W^*, integrating with respect to z from 0 to 1 using the boundary conditions on W and again multiplying (3.21) by W^* and (3.22) by W, integrating with respect to z from 0 to 1 and subtracting the resulting equations, we get

$$(\lambda - \lambda^*)\langle W,\ L_0^* W^* \rangle = 0, \qquad \text{... (3.23)}$$

where the angle brackets $\langle \ldots \rangle$ denote the integration with respect to z from 0 to 1. Since W and W^* are arbitrary, it follows that $\lambda - \lambda^* = 0$, that is λ is real and the eigenfunctions are orthogonal. Hence the Theorem 2.

***Theorem* 3** — *Under the hypothesis of Theorem 2 the eigenvalues are stationary.*

PROOF : Multiplying (3.21) by W and (3.22) by W^*, integrating with respect to z from 0 to 1, using the boundary conditions on W, and using Theorem 2, we get

$$\lambda = \frac{\langle W^*, LW \rangle}{\langle W^*, L_0 W \rangle} \,. \qquad \ldots (3.24)$$

Taking the variation on both sides of (3.24), using (3.21) and (3.22), we obtain

$$\delta\lambda = 0.$$

This implies that λ is stationary. Hence the Theorem 3.

3.3 Relevant Equations for Galerkin Method

Theorems 1 and 2 enables us to use the Galerkin method to obtain the eigenvalues. To get the required equations for the Galerkin method, we set $m = 1$ and $\sigma = 0$ in (3.10) and (3.11), and using the boundary conditions[18], we obtain the equations

$$\langle \left(D^2 W\right)^2 + (2a^2 + Q)\,(DW)^2 + a^4 W^2 \rangle = aR^{1/2} \langle WT \rangle, \qquad \ldots (3.25)$$

and

$$\langle (DT)^2 + a^2 T^2 \rangle = aR^{1/2} \langle f(z)\, WT \rangle. \qquad \ldots (3.26)$$

Substituting $W = AW_1$ and $T = BT_1$ into (3.25) and (3.26), where A and B are constants and W_1 and T_1 are trial functions, and eliminating A and B and omitting, for simplicity, the subscript 1 on the dependent variables, we get

$$R = \frac{\langle (D^2 W)^2 + (2a^2 + Q)\,(DW)^2 + a^4 W^2 \rangle \langle (DT)^2 + a^2 T^2 \rangle}{a^2 \langle WT \rangle \langle f(z)\, WT \rangle} \qquad \ldots (3.27)$$

Since the dependence of the value of the critical Rayleigh number, R_c, on the form of the fluid boundaries is crucial, we consider in this article the isothermal and constant heat flux boundaries.

3.4 Critical Rayleigh Number for Isothermal Boundaries

We consider the three cases : (i) both boundaries rigid, (ii) both boundaries free and (iii) lower boundary rigid and upper boundary free.

3.4.1. Both Boundaries Rigid — The boundary conditions are

$$W = DW = T = 0, \text{ on } z = 0 \text{ and } z = 1. \qquad \ldots (3.28)$$

We select the trial functions

$$W = z^2 (1 - z)^2 \text{ and } T = z(1 - z), \qquad \dots (3.29)$$

satisfying the boundary conditions, (3.28). Substituting (3.29) into (3.27) and performing the integration, we get

$$R = \frac{[x^2 + 12(12x + Q) + 504]\,(10 + X)}{135x\,\langle f(z)\,(z^3 - 3z^4 + 3z^5 - z^6)\,\rangle}\,, \qquad \dots (3.30)$$

where $X = a^2$. For any given $f(z)$, R attains its minimum value, R_c at $X_c = a_c^2$, X_c being the root of

$$X^3 + 17x^2 - (2520 + 60Q) = 0. \qquad \dots (3.31)$$

We note that for large Q, (3.31) tends to the asymptotic value $a_c \to 1.98\,Q^{1/6}$ and the corresponding asymptotic value of R_c, from (3.30), is

$$R_c \to \frac{X_c^2}{135\,\langle f(z)\,(z^3 - 3z^4 + 3z^5 - z^6)\,\rangle}\,. \qquad \dots (3.32)$$

The asymptotic behaviour of R_c depends crucially on the nature of heating but the asymptotic behaviour of a_c is independent of the nature of heating.

The variation of X_c with Q is computed from (3.31) for different values of Q and the results are depicted in Table I. It is clear that X_c increases with increasing Q and hence the effect of magnetic field is to contract the cells. Since R_c depends crucially on the nature of heating (i.e., on $f(z)$), our object is to find a suitable $f(z)$ that optimises R_c. For this purpose, we consider different basic temperature profiles[19] given in section 3.4.

3.4.1(a) Linear Temperature Profile — In this case

$$f(z) = 1 \qquad \dots (3.33)$$

and hence (3.17) takes the form

$$R_1 = \frac{140\,[X^2 + 12(2X + Q) + 504]\,(10 + X)}{135\,X}\,. \qquad \dots (3.34)$$

The critical Rayleigh number, R_{c_1}, is obtained from (3.34) by substituting $X = X_c$. We find that the results obtained from (3.24), using the value of X_c obtained from (3.31), agree well with those of Chandrasekhar[1] for $f(z) = 1$ and for small values of Q (see Table I). For large values of Q (> 10^2) there is a slight deviation in R_c compared to those given in[1]. This may be due to the fact that the trial functions (3.29) are polynomials in z and does not take into account the Hartmann boundary layer effect that exists at the rigid boundaries for large values of Q.

To obtain the more refined values of R_c, we use the higher order Galerkin expansion. We take

$$W = A_i\, W_i \text{ and } T = B_i\, T_i \qquad \text{... (3.35)}$$

where

$$W_i = (1 - z)^2\, z^{i+1} \text{ and } T_i = (1 - z)\, z^i.$$

Substituting (3.35) into (3.25) and (3.26) we get after simplification

$$A_i\, D_{ji} + B_i\, E_{ji} = 0$$

and

$$A_i\, F_{ji} + B_i\, G_{ji} = 0 \qquad \text{... (3.36)}$$

where

$$D_{ji} = \langle D^2 W_j\, D^2 W_i + (2a^2 + Q)\, DW_j\, DW_i + a^4\, W_i\, W_j \rangle ,$$

$$E_{ji} = -a^2 R \langle W_j\, T_i \rangle ,$$

$$F_{ji} = \langle f(z)\, T_j\, W_i \rangle$$

and

$$G_{ji} = \langle DT_j\, DT_i + a^2\, T_j\, T_i \rangle.$$

Non-trivial solution of (3.36) exists if and only if the corresponding characteristic determinant is zero. That is

$$\begin{vmatrix} D_{ji} & E_{ji} \\ F_{ji} & G_{ji} \end{vmatrix} = 0. \qquad \text{... (3.37)}$$

This leads to a relation between the characteristic parameters R, Q and a in the form

$$\Psi(R, a, Q) = 0. \qquad \text{... (3.38)}$$

By fixing Q, we may obtain R_c and a_c. The R_c for different values of Q are obtained numerically by solving (3.24) for different temperature profiles. The critical values obtained for four terms in the Galerkin expansion (3.35) are tabulated in the Table I. We see that for $f(z) = 1$ the results are close to those given by Chandrasekhar[1].

We also see that the asymptotic value of R_{c_1} when $Q \to \infty$, obtained from (3.32) with $a_c \to 1.98\, Q^{1/6}$, is $R_{c_1} \to 12.5\, Q$. Substituting for R_{c_1} and Q in accordance with their definitions, we find that the formula which determines the critical temperature gradient for the onset of stationary convection when $Q \to \infty$ is

$$\alpha g \beta_c \to 12.5 \frac{\mu^2 H_0^2 d^2 K \sigma}{\rho_0}\,, \qquad \text{... (3.39)}$$

where β_c is the critical temperature gradient. From this it is clear that β_c is independent of the kinematic viscosity ν. The physical meaning of this will be discussed in section 3.5.

3.4.1(b) Piece-wise Linear Profile with Heating from Below — In this case, following Currie[15], we have

$$f(z) = \begin{cases} \varepsilon^{-1}, & 0 \le z < \varepsilon \\ 0, & \varepsilon < z \le 1, \end{cases} \qquad \text{... (3.40)}$$

where ε is the thermal depth. Substituting (3.40) into (3.30) and using $X = X_c$, we get

$$R_{c_2} = \frac{R_{c_1}}{\text{Max}\, \varepsilon^2 (35 - 84\varepsilon + 70\varepsilon^2 - 20\varepsilon^3}\,. \qquad \text{... (3.41)}$$

As ε increases from 0 to 1, R_{c_2} decreases from infinity to a minimum value of

$$R_{c_2} = \frac{R_{c_1}}{1.249} \qquad \text{... (3.42)}$$

attained at $\varepsilon = 0.71$ and then increases to R_{c_1} at $\varepsilon = 1$. R_{c_2}, given by (3.42), is computed for different values of Q. The results are shown in Table I and discussed in the final section. When $Q = 0$, (3.41) gives $R_{c_2} = 1401$ which is close to the experimental value $R_{c_2} = 1400$ obtained by DeGraff and Vander Held[20] and is an improved value compared to the value $R_c = 1340$ given by Currie[15]. We see that the value obtained by using Galerkin technique is in agreement with the experimental value.

The asymptotic value of R_{c_2} when $Q \to \infty$ obtained from (3.42) with $a_c \to 1.98\, Q^{2/6}$ is $R_{c_2} \to 10\, Q$. The critical temperature gradient β_c in this case is given by

$$\alpha g \beta_c \to 10 \frac{\mu H_0{}^2 d^2 K \sigma}{\rho_0}\,, \qquad \text{... (3.43)}$$

which is independent of the kinematic viscosity ν. A physical explanation for this will be given in section 3.5.

Similarly the critical Rayleigh number R_{c_3} for piece-wise linear profile with cooling from above given by Vidal and Acrivos[21] in the form

$$f(z) = \begin{cases} 0 & 0 \le z < 1 - \varepsilon \\ 1 & 1 - \varepsilon < z \le 1 \end{cases} \qquad \text{... (3.44)}$$

R_{c_4} for step-function profile

$$f(z) = \delta(z - \varepsilon) \qquad \text{... (3.45)}$$

R_{c_5} for inverted parabolic profile

$$f(z) = \delta(1 - z) \qquad \text{... (3.46)}$$

R_{c_6} for parabolic profile

$$f(z) = 2z \qquad \text{... (3.47)}$$

are computed from (3.32) as in section 3.4.1(a) and 3.4.1(b) and we found that

$$R_{c_3} = R_{c_2}, R_{c_4} = \frac{R_{c_1}}{35}, R_{c_5} = R_{c_6} = R_{c_1}. \qquad \text{... (3.48)}$$

3.4.2. Both Boundaries Free and Isothermal

The boundary conditions are

$$W = D^2 W = T = 0 \text{ at } z = 0, 1.$$

In this case, we select the trial functions.

$$W = z(1 - z)(1 + z - z^2) \text{ and } T = z(1 - z), \qquad \text{... (3.49)}$$

satisfying the above boundary conditions. Substituting (3.41) into (3.14), we get

$$R = \frac{\{31x^2 + 306(2x + Q) + 3024\}(10 + x)}{765x \langle f(z)(z^2 - z^3 - 2z^4 + 3z^5 - z^6) \rangle}. \qquad \text{... (3.50)}$$

For a given $f(z)$, this attains a minimum value R_c at $X = X_c$, X_c being the root of

$$X^3 + 14.87X^2 - (487.74 + 49.35\,Q) = 0 \qquad \text{... (3.51)}$$

For large Q, (3.51) gives the asymptotic value $a_c \rightarrow 1.91\,Q^{1/6}$. Hence, the asymptotic value of R, from (3.50) for large Q is

$$R \rightarrow \frac{x^2}{24.6 \langle f(z)\, WT \rangle}. \qquad \text{... (3.52)}$$

The critical wave number X_c and the corresponding Rayleigh number R_{c_i} ($i = 1$ to 6) are computed for different values of Q and the basic temperature profiles given in section 3.4.1. The results are shown in Table I. We see that, compared with rigid boundaries, free boundaries elongate the cells and augment convection, as expected on physical grounds.

3.4.3. Lower Boundary Rigid and Upper Boundary Free — The boundary conditions are

$$W = DW = T = 0 \text{ at } z = 0$$
$$W = D^2 W = T = 0 \text{ at } z = 1. \quad \text{... (3.53)}$$

These are satisfied by

$$W = z^2 (1 - z)(3 - 2z) \text{ and } T = z(1 - z). \quad \text{... (3.54)}$$

In this case (3.14) takes the form

$$R = \frac{\{19x^2 + 216(2x + Q) + 4536\}(10 + x)}{585X \langle f(z)(3z^3 - 8z^4 + 7z^5 - 2z^6)\rangle}, \quad \text{... (3.55)}$$

and attains the minimum value R_c at $X = X_c$. Where X_c, the critical wave number, is a root of

$$X^3 + 16.37X^2 - (1193.68 + 56.84\, Q) = 0. \quad \text{... (3.56)}$$

For large Q, (3.56) tends to the asymptotic value $a_c \to 1.96\, Q^{1/6}$ and the corresponding asymptotic value of R from (3.55) is

$$R_c \to \frac{X_c^2}{30.78\langle f(z)WT\rangle}. \quad \text{... (3.57)}$$

The X_c and the corresponding R_{c_i} (i = 1 to 6) are computed for different values of Q and the basic temperature profiles given in section 3.4.1. The results are shown in Table II and are discussed in the final section.

3.5 Conditions for Onset of Convection in an Inviscid Electrically Conducting Fluid, in the presence of Transverse Magnetic Field and Non-uniform Temperature Gradient

In the case of stress free boundaries with uniform temperature gradient Chandrasekhar[1] has given the origin of $\pi^2 Q$ law using the inviscid theory. In this section we extend his analysis to non-uniform temperature gradients with the object of establishing suitable asymptotic value of R_c.

The asymptotic values of R_{c_i} (i = 1 to 6) when $Q \to \infty$ discussed in sections 3.4.1. to 3.4.3 for different boundary combinations and for different non-uniform temperature gradients for the onset of convection become independent of the coefficient of viscosity ν, but depends on H_0, σ and $f(z)$. R_{c_i} become independent of ν when $Q \to \infty$ (i.e., when $H_0 \to \infty$, $\sigma \to \infty$ or $\nu \to 0$) mainly because the dissipation of energy by viscosity and as a consequence the marginal stability is essentially determined by the equality of these energies[1]. This can be established by considering the equations relevant for an inviscid finitely conducting fluid.

The basic equations for the onset of stationary convection, from (3.1) to (3.4) with $\nu \to 0$ and using the normal mode analysis, are

$$0 = aR^{1/2}T - \frac{Q}{P_m}D(D^2 - a^2)H, \quad \text{... (3.58)}$$

$$0 = aR^{1/2} fW + (D^2 - a^2)\, T \qquad \text{... (3.59)}$$

and

$$0 = -P_m DW + (D^2 - a^2)\, H. \qquad \text{... (3.60)}$$

Eliminating H between (3.58) and (3.60), we get

$$0 = aR^{1/2}\, T + QD^2\, W. \qquad \text{... (3.61)}$$

Applying the Galerkin procedure as explained in the previous sections, we get

$$R = \frac{Q \langle (DT)^2 + a^2 T^2 \rangle \langle (DW)^2 \rangle}{a^2 \langle fWT \rangle \langle WT \rangle}. \qquad \text{... (3.62)}$$

Minimum value of R, denoted by R_c, occur for $a \to \infty$ and is given by .

$$R_c \to \frac{Q \langle T^2 \rangle \langle (DW)^2 \rangle}{\langle f wT \rangle \langle WT \rangle}. \qquad \text{... (3.63)}$$

As in the previous sections here also we consider the three boundary combinations, viz, Rigid-Rigid, Free-Free and Rigid-Free boundaries.

3.5.1. Both Boundaries Rigid and Isothermal

In this case (3.63) using the trial functions (3.16) takes the form

$$R_c = \frac{0.0888867\, Q}{\langle f(z)\,(z^3 - 3z^4 + 3z^5 - z^6) \rangle}. \qquad \text{... (3.64)}$$

For different temperature profiles given in section 3.4.1 and following the analysis of section 3.4.1 we get

$$R_{c_1} \to 12.44Q,\ R_{c_2} = R_{c_3}\ 9.97Q \text{ attained at } \varepsilon = 0.71 \qquad \text{... (3.65)}$$

$$R_{c_4} \to 5.71Q \text{ attained at } \varepsilon = 0.5,\ R_{c_5} = R_{c_6} = 12.44Q,$$

which give the correct asymptotic behaviour when both boundaries are rigid and isothermal.

3.5.2 Both Boundaries Free and Isothermal

In this case (3.63), using trial functions (3.49), takes the form

$$R_c = \frac{0.4000004\, Q}{\langle fWT \rangle}. \qquad \text{... (3.66)}$$

For different temperature profiles given in section 3.4.1 and following the analysis of section 3.4.2 we get $R_{c_1} \to 9.88Q$, $R_{c_2} = R_{c_3} \to 8.19\ Q$ attained at ε = 0.72.

$$R_{c_4} \to 5.12\ Q \text{ attained at } \varepsilon = 0.5,\ R_{c_5} = R_{c_6} \to 9.88\ Q, \qquad \text{... (3.67)}$$

which are the correct asymptotic behaviour when both boundaries are free and isothermal.

3.5.3. Lower Boundary Rigid Isothermal and Upper Boundary Free Isothermal

In this case (3.63), using the trial functions (3.54), takes the form

$$R_c = \frac{0.3692303\, Q}{\langle fWT \rangle}. \qquad \text{... (3.68)}$$

For different temperature profiles given in section 3.4.1 and following the analysis of section 3.4.3. we get

$$R_{c_1} \to 11.93Q,\ R_{c_2} \to 10.20\ Q \text{ attained at } \varepsilon = 0.76,$$

$$R_{c_3} = 9.18\ Q \text{ attained at } \varepsilon = 0.65$$

$$R_{c_4} \to 5.77\ Q \text{ attained at } \varepsilon = 0.54$$

$$R_{c_5} \to 12.92\ Q,\ R_{c_6} \to 11.08\ Q \qquad \text{... (3.69)}$$

which are the correct asymptotic behaviour in this case.

In all the boundary combinations the critical wave number is independent of the nature of heating and in this inviscid limit the cells are infinitely narrow. Further, in all the three boundary combinations, the critical temperature at which marginal instability sets in is given by

$$\alpha g \beta_c = \text{Constant } \nu_{eff} K d^{-4}, \qquad \text{... (3.70)}$$

where

$$\nu_{eff} = \frac{\sigma}{\rho} (\mu H d)^2. \qquad \text{... (3.71)}$$

The constant (in 3.70) takes different values for different boundary combinations (which is in contrast to the results of Chandrasekhar[1]) and for different basic temperature profiles. Eq. (3.71) reveals that the presence of the magnetic field imparts to the liquid an effective kinematic viscosity ν_{eff}.

The above results are obtained for $\nu \to 0$. To obtain the correct dependence of a on Q, we have to retain a^4W in (3.5) in addition to QD^2W. In other words, in place of (3.61) we have to use the equation

$$0 = aR^{1/2}\, T + (QD^2 - a^4)W. \qquad \text{... (3.72)}$$

From (3.59) and (3.72), following the procedure explained above, in place of (3.62) we get

$$R = \frac{\langle (DT)^2 + a^2T^2 \rangle \langle Q\,(DW)^2 + a^4W^2 \rangle}{a^2 \langle fWT \rangle \langle WT \rangle}. \qquad \text{... (3.73)}$$

This equation is sufficient to predict the correct asymptotic behaviours of both R_{c_i} (i = 1 to 6) and a_c. Following the same procedure described above we get $a_c \to 1.98\ Q^{1/6}$ when $Q \to \infty$. The presence of viscosity prevents the cells from collasping into lines.

4. Critical Rayleigh Number for Constant Heat Flux at the Boundaries

As in the previous sections, here also we find the condition for the onset of convection in a layer with free-free or rigid-rigid boundaries specifying constant heat flux (hereafter called adiabatic boundaries) at the boundaries. We consider here the transient heating because in the case of adiabatic boundaries steady state basic linear temperature profiles are not feasible.

4.1. *Free-Free Adiabatic Boundaries*

In this case the required boundary conditions are

$$W = D^2W = DT = 0 \quad \text{at } z = 0,\ 1. \qquad \text{... (4.1)}$$

The trail functions satisfying these conditions are

$$W = z(1-z)(1+z-z^2) \text{ and } T = 1. \qquad \text{... (4.2)}$$

Substituting these in (3.27), we get

$$R = \frac{3024 + (2x + Q)306 + 31x^2}{126\,\langle f(z)\, WT\rangle}, \qquad \text{... (4.3)}$$

which attains the minimum value R_c at a = 0. Hence the critical Rayleigh number for a uniform temperature gradient $f(z)$ = 1, from (4.3) is

$$R_{c_1} = 120 + 12.14Q. \qquad \text{... (4.4)}$$

In the absence of magnetic field (Q = 0), (4.4) gives R_{c_1} = 120, which is the known value. When $Q \to \infty$ $R_{c_1} \to 12.14Q$ in contrast $\pi^2 Q$ in the isothermal case of Chandrasekhar[1].

The critical Rayleigh numbers R_{c_i} (i = 1 to 6) are computed as in section 3 for different temperature profiles and we found,

$$R_{c_2} = 105.56 + 10.68Q,\ \varepsilon = 0.743,$$

$$R_{c_3} = R_{c_2},\ R_{c_4} = 76.8 + 7.771430,\ \varepsilon = 0.5,\ R_{c_5} = R_{c_6} = R_{c_1} \qquad \text{... (4.5)}$$

4.2. *Adiabatic Rigid Lower Boundary and Adiabatic Free Upper Boundary*

The Boundary conditions are now

$$W = DW = DT = 0 \quad z = 0 \qquad \text{... (4.6)}$$

$$W = D^2W = DT = 0 \quad z = 1.$$

These are satisfied by

$$W = z^2(1-z)(3-2z) \text{ and } T = 1. \qquad \text{... (4.7)}$$

The critical Rayleigh number occurs at $a = 0$. Then from (3.14) using (4.7) and $a = 0$, we get

$$R_c = \frac{9072 + 432Q}{189\langle f(z)WT\rangle}. \qquad \text{... (4.8)}$$

For $f(z) = 1$, we obtain

$$R_{c_1} = 320 + 15.24Q. \qquad \text{... (4.9)}$$

When $Q \to 0$, $R_{c_1} \to 320$ which is the known value. However, when $Q \to \infty$, $R'_{c_1} \to 15.24\,Q$ which is higher than the true value $12Q$. This may be due to the fact that the polynomial trial functions (4.7) does not take into account the Hartmann boundary layer that exists at the rigid boundary. This aspect will be considered in sections 5 and 6. For temperature profiles given by the section 3.4.1, we obtain

$$R_{c_2} = 292.61 + 13.93Q,\ \varepsilon = 0.821,\ R_{c_3} = 251.96 + 12Q,\ \varepsilon = 0.638, \qquad \text{... (4.10)}$$

$$R_{c_4} = 184.63 + 8.79Q,\ \varepsilon = 0.538,\ R_{c_5} = 360 + 17.14Q,$$

$$R_{c_6} = 288 + 13.71Q$$

4.3. Adiabatic Rigid Boundaries

In this case the boundaries are

$$W = DW = DT = 0 \quad \text{at} \quad z = 0,\ 1. \qquad \text{... (4.11)}$$

The trial functions satisfying these conditions are

$$W = z^2(1-z)^2 \text{ and } T = 1. \qquad \text{... (4.12)}$$

Substituing these in (3.27) and using $a = 0$, we get the critical Rayleigh number

$$R_c = \frac{504 + 12Q}{21\langle f(z)\,WT\rangle}. \qquad \text{... (4.13)}$$

For different temperature profiles given in section 3.4.1. and following the analysis of previous sections we found that

$$R_{c_1} = 720 + 17.14Q,\ R_{c_2} = R_{c_3} = 601.12 + 14.31Q,\ \varepsilon = 0.724, \qquad \text{... (4.14)}$$

$$R_{c_4} = 384 + 9.14Q,\ \varepsilon = 0.5,\ R_{c_5} = R_{c_6} = R_{c_1}$$

When $Q \to 0$, $R_{c_1} \to 720$ which is the known value, when $Q \to \infty$, $R_{c_1} \to 17.14Q$ which is higher than the known value $12Q$ because the polynomial type of trial functions (4.17) does not take care of the Hartmann boundary layer that exists for large values of Q.

4.4. *Critical Rayleigh Number for an Isòthermal Rigid Lower Boundary and an Adiabatic Free Upper Boundary*

In the previous sections we have been concerned with deriving the condition for the onset of magnetoconvection either with a constant temperature (i.e., isothermal) or constant heat flux (i.e. adiabatic) at the boundaries. In the case of the free boundaries specifying a constant temperature is more restrictive than specifying a constant heat flux. Therefore, in this section we find R_c when the layer is bounded below by an isothermal rigid boundary and above by an adiabatic free boundary. The required boundary conditions are

$$W = DW = T = 0, \; z = 0 \qquad \text{... (4.15)}$$

$$W = D^2W = DT = 0, \; z = 1.$$

These are satisfied by

$$W = z^2(1-z)(3-2z) \text{ and } T = z(1 - z/2). \qquad \text{... (4.16)}$$

Equation (3.14) then yields

$$R = \frac{\{19x^2 + 2.6\,(2x + Q) + 4536\}\,(10 + x)}{585x\,\langle f(z)\, WT \rangle}, \qquad \text{... (4.17)}$$

which attains the minimum value, for any $f(z)$, R_c at $X = X_c$, X_c being the root of

$$X^3 + 16.37X^2 - (1193.68 + 56.84Q) = 0. \qquad \text{... (4.18)}$$

For large Q, (4.18) tends to the asymptotic value $a_c = X_c^{1/2} = 1.96\, Q^{1/6}$ and the corresponding asymptotic value of R is

$$R_c \to \frac{0.06X^2}{\langle f(z)\,(6Z^3 - 13Z^4 + 9Z^5 - 2Z^6)\rangle}. \qquad \text{... (4.19)}$$

For $f(z) = 1$, the asymptotic value of R i.e., $R_c \to 11.9\, Q$ in contrast to $\pi^2 Q$ for isothermal boundaries[1].

The values of X_c and R_{c_i} (i = 1 to 6) are computed as in the previous sections, for different values of Q and the results are depicted in table. In this case $R_{c_3} \neq R_{c_2}$ and $R_{c_1} \neq R_{c_5} \neq R_{c_6}$ because of the absence of symmetry in WT. We note that as ε increases from 0 to 1, R_{c_2} decreases from infinity to a minimum value $R_{c_2} = R_{c_1} / 1.069$ attained at $\varepsilon = 0.86$ and R_{c_3} decreases to a minimum value.

$$R_{c_3} = R_{c_1}/1.423 \text{ attained at } \varepsilon = 0.55.$$

5. Regular Perturbation Technique

The analysis of section 4 reveals that a single-term Galerkin expansion gives known results for the case of free-free adiabatic boundaries but the results deviate from the known values in the case of rigid-rigid adiabatic boundaries. For example in the special case of linear temperature profile, (4.13) gives $R_{c_1} \rightarrow 17.14\ Q$ for rigid-rigid boundaries when $Q \rightarrow \infty$ in contrast to the known value $R_{c_1} \rightarrow 12\,Q$. The same is also true in the case of (4.7) and (4.15). This inaccuracy may be attributed to the fact for large values of Q, Hartmann boundary layer inevitably arises at the rigid boundary and the polynomial type of trial function in a single term Galerkin expansion does not take care of this boundary layer. Hence, a single term Galerkin expansion technique is not suitable to determine R_c. In this section, we therefore propose the use of a regular perturbation technique with the wave number 'a' as a parameter to obtain exact solutions of the stability equations (3.5) to (3.70) with $\sigma = 0$. From this exact solution we can determine R_{c_i} for different temperature profiles.

We first eliminate H between (3.5) and (3.7) and obtain

$$(D^2 - a^2)W - QD^2W = a^2RT, \qquad \text{... (5.1)}$$

and

$$(D^2 - a^2)T = -f(z)w. \qquad \text{... (5.2)}$$

Since 'a' is small in the case of specifying constant heat flux at the boundaries, we look for solutions of (5.1) and (5.2) in the form

$$(W,\ T) = (W_0, T_0) + a^2(W_1, T_1) + \ldots.$$

Physically, the small wave number means that the lateral extent of a single preferred mode of convection cell is limited only by the presence of lateral walls. This phenomenon was discussed in detail by Nield[22].

Substituting (5.3) into (5.1) and (5.2) and equating the coefficients of the like powers in a^2, we get

$$D^4W_0 - QD^2W_0 = 0, \quad D^2T_0 = -f(z)W_0, \qquad \text{... (5.4)}$$

$$D^4W_1 - QD^2W_1 = RT_0 + 2D^2W_0 \qquad \text{... (5.5)}$$

and

$$D^2T_1 = T_0 - f(z)W_1. \qquad \text{... (5.6)}$$

These equations have to be solved subject to the boundary conditions

$$W_1 = DW_1 = DT_1 = 0 \ (i = 0,\ 1) \text{ at } z = 0,\ 1, \qquad \text{... (5.7)}$$

in the case of rigid-rigid boundaries with constant heat flux and

$$W_i = D^2W_i = DT_i = 0 \quad (i = 0,\ 1) \text{ at } z = 0,\ 1, \qquad \text{... (5.8)}$$

in the case of stress free boundaries with constant heat flux.

For any combination of boundaries (i.e., rigid-rigid, free-free or rigid-free), the solutions of (5.4) is

$$W_0 = 0, \text{ and } T_0 = 1. \qquad \dots (5.9)$$

The general solutions of (5.5) and (5.6) are

$$W_1 = R\left[\frac{1}{2Q} z^2 + C_0 + C_1 z + C_2 e^{Q_1 z} + C_3 e^{-Q_1 z}\right] \qquad \dots (5.10)$$

and $$T_1 = \frac{z^2}{2} + t_1 z + t_0 - \int\int f(z)\, W_1(z)\, dz, \qquad \dots (5.11)$$

where $Q_1 = \sqrt{Q}$.

Knowing $f(z)$, from the nature of heating, we can find $T(z)$.

For example, for linear temperature distribution $f(z) = 1$, we have

$$T_1 = \frac{z^2}{2} + t_1 z + t_0 - R$$

$$\left[\frac{1}{2Q}\frac{z^4}{12} + C_0 \frac{z^2}{2} + C_1 \frac{z^3}{6} + C_2\, Q_1^2\, e^{Q_1 z} + C_3\, Q_1\, e^{-Q_1 z}\right], \qquad \dots (5.12)$$

and for parabolic distribution, $f(z) = 2z$, we obtain

$$T_1 = \frac{z^2}{2} + t_1 z + t_0 + \ 2R\left[\frac{z^5}{40Q} + \frac{C_1 z^4}{12} + C_0 \frac{z^3}{6}\right.$$

$$\left. + \frac{C_2}{Q_1^2}(Q_1 z - 2)\, e^{Q_1 z} + C_3\, z \frac{e^{-Q_1 z}}{Q_1^2}\right]. \qquad \dots (5.13)$$

From (5.6), being the constant heat flux condition, we obtain

$$\int_0^1 \{\, 1 - f(z)\, W_1 \,\}\ dz = \int_0^1 D^2\, T_1 dZ = 0. \qquad \dots (5.14)$$

This, using (5.10), becomes

$$R = \frac{-1}{\int_0^1 f(z)\, W_1\, dZ}. \qquad \dots (5.15)$$

To determine R, we should find the constants C_i (i = 0 to 3) in (5.10) which depend on the nature of the boundaries. As before, we consider the three boundary combinations : (i) both boundaries rigid, (ii) both boundaries free and (iii) lower boundary rigid and upper boundary free.

5.1. Both Boundaries Rigid

The boundary conditions from (5.7) are

$$W_1 = DW_1 = DT_1 = 0 \quad \text{at} \quad z = 0,\ 1. \qquad \dots (5.16)$$

Then from (5.10), using (5.16), we get

$$C_0 = \frac{Q_1 \cosh Q_1 - 2 \sinh Q_1 + Q_1}{Q Q_1 \Delta} ,$$

$$C_1 = \frac{2 \cosh Q_1 - Q_1 \sinh Q_1 - 2}{Q \Delta} , \qquad \ldots (5.17)$$

$$C_2 = \frac{(2 - Q_1 - Q_1 e^{Q_1} - 2 e^{-Q_1})}{2 Q Q_1 \Delta} ,$$

$$C_3 = \frac{2e^{Q_1} - Q_1 - Q_1 e^{Q_1} - 2}{2 Q Q_1 \Delta} ,$$

$$\Delta = 2(2 + Q_1 \sinh Q_1 - 2 \cosh Q_1).$$

Our object, as before, is to find a suitable temperature profile that optimises R. For this purpose we consider different basic temperature profiles discussed in section 3.4.1.

Model 1. Linear Basic Temperature Distribution

In this model

$$f(z) = 1. \qquad \ldots (5.18)$$

Substituting (5.10), (5.11) and (5.18) into (5.15) and simplifying, we get

$$R_{c_1} = \frac{6Q^2 (4 + 2Q_1 \sinh Q_1 - 4 \cosh Q_1)}{(24 - 4Q) + (QQ_1 + 24Q_1) \sinh Q_1 - (8Q + 24) \cosh Q_1} . \qquad \ldots (5.19)$$

When $Q \to 0$, (5.19) gives the critical value $(R_C)_1 = 720.0$ which is the known exact value in the absence of magnetic field (Nield 1975)[23]. Similarly, the asymptotic value of R_{c_1} for large Q is

$$R_{c_1} \to 12Q \text{ as } Q \to \infty \qquad \ldots (5.20)$$

in contrast to the value $R_{c_1} \to 17.14Q$ given by the single term Galerkin expansion technique used in section 4 (see eq. 4.14). The R_{c_1} given by (5.20) is close to the value $R_c \to 12.44Q$ obtained for isothermal boundaries using the single term Galerkin expansion procedure. From this we conslude that the critical Rayleigh number in the case of uniform temperature gradient with rigid boundaries, isothermal or adiabatic, obeys $12Q$ instead of $\pi^2 Q$ law concluded by Chandrasekhar[1].

The critical Rayleigh number R_{c_1} is computed from (5.19) for different values of Q and the results are depicted in Table IV.

Model 2. Piece-wise Linear Profile With Heating from Below

In this case $f(z)$ is given by (3.40). Substituting (3.40) and (5.10) into (5.15) we get

$$R_{c_2} = \frac{6Q^2 \varepsilon(4 \cosh Q_1 - 2Q_1 \sinh Q_1 - 4)}{\Delta_1}, \qquad \ldots (5.21)$$

where

$$\Delta_1 = a_0 + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3$$

$$a_0 = 3\ \{4 \cosh (Q_1\varepsilon) + 4 \cosh Q_1 - 4 \cosh (Q_1\varepsilon - Q_1)$$

$$- 2Q_1 \sinh (Q_1\varepsilon - Q_1) - 2Q_1 \sinh (Q_1\varepsilon) - 2Q_1 \sinh Q_1 - 4$$

$$a_1 = 6\ (Q_1 - 2Q_1 \sinh Q_1 + Q \cosh Q_1)$$

$$a_2 = 3Q\ (2 \cosh Q_1 - Q_1 \sinh Q_1 - 2)$$

$$a_3 = 2Q\ (2 + Q_1 \sinh Q_1 - 2 \cosh Q_1).$$

when $Q \to 0$, (5.21) revelas that as ε increases from 0 to 1, R_{c_2} decreases from infinity to a minimum value $R_{c_2} = 601.1$ attained at $\varepsilon = 0.75$ and then increases to R_{c_1} at $\varepsilon = 1$.

$$R_{c_2} \to 10.7Q \text{ when } Q \to \infty, \qquad \ldots (5.22)$$

in contrast to the value 14.31 Q given by (4.14) using the Galerkin method. This value of R_{c_2} given by (5.22) is close to the value $R_{c_2} \to 10\,Q$ for isothermal boundaries (see equation 3.30). R_{c_2} given by (5.21) is calculated for different values of Q and the values are tubulated in Table IV.

Model 3. Piece-wise Linear Profile with Cooling from above

In this case, the expression for $f(z)$ is given by (3.44). Substituting (3.44) and (5.10) into (5.15) we find, as in section 4, that

$$R_{c_3} = R_{c_2} \qquad \ldots (5.23)$$

This equality is due to the symmetry in *WT* about the mid-line.

Model 4. Step Function Temperature Profile

In this model, we consider the step function profile given by (3.45). Substituting (3.45) and (5.10) into (5.15), we get

$$R_{c_4} = \frac{4Q^2 (\sinh Q_1 - Q_1 \cosh Q_1)}{C_1 \cosh Q_1 + C_2 \sinh Q_1 + C_3 + C_4}, \qquad \ldots (5.24)$$

where

$$C_1 = 24QQ_1\, \varepsilon^2 + 4Q_1\, \varepsilon - 2QQ_1\, \varepsilon,$$

$$C_2 = 2Q - 2Q\, \varepsilon^2 - 4$$

$$C_3 = 2Q \sinh(Q_1 \varepsilon - Q_1) - 4 \sinh(Q_1 \varepsilon - Q_1) \\ -4\ Q_1 \cosh(Q_1 \varepsilon) + 4 \sinh(Q_1 \varepsilon)$$

and

$$C_4 = 4Q_1 - 4Q_1 \varepsilon.$$

When $Q \to 0$, (5.24) yields the critical value $R_{c_4} = 184.63$ attained at $\varepsilon = 0.578$. As in the earlier case R_{c_4} decreases from infinity to a minimum value at $\varepsilon = 0.578$ and increases again at $\varepsilon = 1$. The asymptotic value R_{c_4} for large value of Q is

$$R_{c_4} \to 8.0\, Q \text{ when } Q \to \infty, \qquad \dots (5.25)$$

in contrast to the value 9.14 Q given by (4.14). R_{c_4} given by (5.24) is calculated for different values of Q and the results are depicted in Table IV.

Model 5. Inverted parabolic Temperature Profile

The inverted parabolic temperature is given by (3.46). Substituting (3.46) and (5.10) into (5.15) we find that

$$(R_c)_5 = (R_c)_1.$$

Model 6. Parabolic Temperature Profile

The parabolic temperature profile is given by (3.47). Substituting (3.47) and (5.10) into (5.15), we observe that

$$(R_c)_6 = (R_c)_1.$$

It is interesting to note that when the boundaries are symmetric, then the effect of parabolic and inverted parabolic temperature profiles on the onset of convection is the same as the linear profile.

5.2. Lower Boundary Rigid and Upper Boundary Free

The boundary conditions are given by (4.6). In this case the constants in (5.10) take the expressions

$$C_0 = \frac{Q \sinh Q_1 + 2Q_1 - 2 \sinh Q_1}{\Delta Q}$$

$$C_1 = \frac{2 \cosh Q_1 - Q \cosh Q_1 - 2}{\Delta Q_1}$$

$$C_2 = \frac{2 - 2Q_1 + Qe^{-Q_1} - 2e^{-Q_1}}{2Q\Delta}$$

$$\Delta = 2Q(Q_1 \cosh Q_1 - \sinh Q_1). \qquad \dots (5.26)$$

For a linear temperature profile (see sec. 5.1, model 1) (5.15), using (5.10) and (5.26) yields

$$R_{c_1} = \frac{6QQ_1(2Q\sinh Q_1 - 2QQ_1 \cosh Q_1)}{(4Q-24)\, Q_1 \sinh Q_1 + (24 - Q^2) \cosh Q_1 + 12(Q-2)}. \qquad \text{... (5.27)}$$

When $Q \to 0$, (5.27) gives $(R_c)_1 = 320.00$ which is the known value. The asymptotic value $(R_c)_1$ for large Q is

$$(R_c)_1 \sim 12\, Q \text{ When } Q \to \infty.$$

$(R_c)_1$ given by (5.27) is computed for different values of Q and are tabulated in the Table IV. Here also we see that R_{c_1} obeys $12Q$ law rather than $\pi^2 Q$ law.

The other temperature profiles are considered similarly as in the section 5.1 and the corresponding critical Rayleigh numbers are computed for different values of Chandrasekhar number Q and are shown in the Table IV and the results are discussed in the final section.

5.3. Both the Boundaries are Free

In this case the required boundary conditions are given by (4.1). Equation (5.10) using (4.1) yields

$$C_0 = -\frac{2\sinh Q_1}{\Delta}$$

$$C_1 = \frac{Q\sinh Q_1}{\Delta}$$

$$C_2 = \frac{(1 - e^{-Q_1})}{\Delta}$$

$$C_3 = \frac{e^{Q_1} - 1}{\Delta}$$

$$\Delta = -2Q^2 \sinh Q_1. \qquad \text{... (5.28)}$$

For linear temperature gradient (Sec. 3.1, model 1) (5.15), using (5.10) and (5.28) gives

$$R_{c_1} = \frac{12Q^2 Q_1 \sinh Q_1}{QQ_1 \sinh Q_1 + 6(4\cosh Q_1 - 4) - 12Q_1 \sinh Q_1}. \qquad \text{... (5.29)}$$

The above equation yields the critical $(R_c)_1 = 120.00$ in the absence of magnetic field $(Q \to 0)$ which coincides with the known value. The asymptotic value of R_{c_1} for large value of Q is

$$R_{c_1} \sim 12Q.$$

R_{c_1} given by (5.29) is computed for different value of Q and the results are shown in the Table IV.

The critical Rayleigh numbers R_{c_i} (i = 2 to 6) for other temperature profiles are computed, as in section 3.1, for different values of Q and the results are tabulated in Table IV and are discussed in the final section.

6. Critical Rayleigh Number Using a Singular Perturbation Technique

To know quantitatively the effect of Hartmann boundary layer that exists at the rigid boundaries for large values of Q on R_c, we find in this section the solutions of (5.5) and (5.6) using a singular perturbation technique (see[24, 19]).

For this purpose we rewrite (5.5) and (5.6) in the form

$$\frac{1}{Q} D^4 W_1 - D^2 W_1 = R_{mc} \qquad \text{... (6.1)}$$

$$D^2 T_1 = 1 - f(z)\, W_1. \qquad \text{... (6.2)}$$

The corresponding boundary conditions are

$$W_1 = DW_1 = DT_1 = 0 \text{ at } z = 0,\ 1. \qquad \text{... (6.3)}$$

Here $R_{mc} = R \,/\, Q.$

The outer solution $W^{(0)}$ of (6.1) for large Q, satisfying

$$D^2 W_1 = R_{mc} \qquad \text{... (6.4)}$$

and the symmetry condition

$$DW^{(0)} = 0 \text{ at } z = 1/2,$$

is

$$W^{(0)} = -R_{mc}\left(C_0 - \frac{1}{2} z + \frac{1}{2} z^2 \right) \qquad \text{... (6.5)}$$

For the region near $z = 0$, the linear solution of (6.1) is of the form

$$W^{(i)} = \chi(Q_1) g(\zeta), \qquad \text{... (6.6)}$$

where $\zeta = Q_1 z$ and $Q_1 = Q^{1/2}$, is the Hartmann number.

The function g satisfies the equation

$$\frac{d^4 g}{d\zeta^4} - \frac{d^2 g}{d\zeta^2} = 0, \qquad \text{... (6.7)}$$

together with the condition

$$g = \frac{dg}{d\zeta} = 0 \text{ at } \zeta = 0,$$

obtained from (6.3).

Since g cannot grow exponentially as $\zeta \to \infty$, we have

$$g = A(1 - \zeta - e^{-\zeta}). \qquad \text{... (6.8)}$$

The matching condition[24]

$$\{W^{(0)}\}_{z \to 0} \text{ matches } \{W^{(i)}\}_{z \to \infty} \qquad \text{... (6.9)}$$

demands that

$$\gamma = \frac{1}{Q_1}, C_0 = 0, A = -\frac{R_{mc}}{2}. \qquad \text{... (610)}$$

Then we have

$$W^{(0)} = \frac{R_{mc}}{2}(z - z^2), \qquad \text{... (6.11)}$$

$$W^{(i)} = -\frac{R_{mc}}{2}\left\{ z - z^2 + \frac{1}{Q_1}(e^{-\zeta} - 1) \right\}. \qquad \text{... (6.12)}$$

Thus, the composite solution, valid near $z = 0$, is of the form

$$W^{(c)} = \frac{R_{mc}}{2}\left\{ z - z^2 + \frac{1}{Q_1}(e^{-Q_1 z} - 1) \right\}. \qquad \text{... (6.13)}$$

Adding the similar contribution from the boundary layer near $z = 1$, we get the uniformly valid solution, of the form

$$W_1 \approx \frac{R_{mc}}{2}\left[Z - Z^2 + \frac{1}{Q_1}\{ e^{Q_1(z-1)} + e^{-Q_1 z} + 1 \} \right]. \qquad \text{... (6.14)}$$

This coincides with (5.15) for large Q.

For a linear temperature profile (see sec. 5.1), (6.14) together with (5.14) yields

$$R_{mc_1} = \frac{12Q}{Q + 12(1 - \cosh Q_1 + \sinh Q_1) - 6Q_1}. \qquad \text{... (6.15)}$$

As $Q1 \to \infty$ (6.15) gives

$$R_{mc_1} \sim 12Q + 72Q^{1/2} \ldots \qquad \text{... (6.16)}$$

The second term here is the effect of Hartmann boundary layer of order $Q^{1/2}$, and its effect is to raise the value of R_{c_1} by an amount of $72Q^{1/2}$. This is due to the fact that QC_0 in (6.5) does not vanish as $Q \to \infty$. The power law in this case is given by $R_{c_1} \sim 12Q$ as $Q \to \infty$ which is close to $\pi^2 Q$ given by Chandrasekhar[1].

For a piece-wise linear profile heating from below (see sec. 5.1), (6.14), gives

$$R_{mc_2} = \frac{12Q\varepsilon}{[Q(3\varepsilon^2 - 2\varepsilon^3)] - 6\{\cosh Q_1 - \sinh Q_1\}\{1 - \sinh(Q_1\varepsilon) - \cosh(Q_1\varepsilon) + \delta_1\}}, \quad \text{... (6.17)}$$

where $\delta_1 = b\{1 + \sinh(Q_1\varepsilon) - \cosh(Q_1\varepsilon) - 6Q_1\}$.

The asymptotic value of R_{mc} as $Q_1 \to \infty$ is

$$R_{mc_2} \sim 10.7Q + 56.88\ Q^{1/2}. \quad \text{... (6.18)}$$

The second term here is the effect of Hartmann boundary layer on the onset of convection. This power is different from the one obtained in section 5.1.

For a piece-wise linear profile cooling from above (see sec. 5.1) we find as before, that

$$R_{mc_3} = R_{mc_2}. \quad \text{... (6.19)}$$

For a step function temperature profile (see sec. 5.1) equation (5.14) using (6.14), gives

$$R_{mc_4} = \frac{2Q_1}{Q_1(\varepsilon - \varepsilon^2) + \frac{\delta_1}{6} - 6Q_1 - (\cosh Q_1 - \sinh Q_1)\{\sinh(Q_1\varepsilon) + \cosh(Q_1\varepsilon)\}}. \quad \text{... (6.20)}$$

Equation (6.20) yields the asymptotic value

$$R_{mc_4} \sim 8Q + 32Q^{1/2} \text{ when } Q \to \infty. \quad \text{... (6.21)}$$

Here also the second term is the contribution from the Hartmann boundary layer. This power law is different from the one obtained in the section 5.1.

For inverted parabolic and parabolic temperature profiles (see sec. 5.1) we find that

$$R_{mc_5} = R_{mc_6} = R_{mc_1}, \quad \text{... (6.22)}$$

due to the symmetry of W_1T_1 about the midline.

7. CONCLUSIONS

The effects of Lorentz force and nonlinear basic temperature profiles on the criterion for the onset of steady marginal magnetoconvection have been established analytically and showed that

(i) a suitable nonuniform temperature gradient and magnetic field suppress or augument magnetoconvection. The extent of suppression depends on the Chandrasekhar number Q, and on a suitable temperature gradient $f(z)$,

(ii) the single term Galerkin expansion procedure provides a quick and easy method for obtaining the critical Rayleigh number R_c, for steady marginal stability and gives results close to the available experimental data compared to the method adopted by Chandrasekhar[1],

(iii) regular and singular perturbation techniques employed in this article predict the effect of Hartmann boundary layer on the onset of magnetoconvection, as the polynomial type of trial function used in the Galerkin expansion procedure does not predict the Hartmann type of boundary layer that exists at the rigid boundaries,

(iv) the asymptotic dependence of the critical Rayleigh number R_c and wavenumber a_c when $Q \to \infty$ obey the power law

$$R \to \text{constant } Q$$

$$a \to \text{constant } Q^{1/6}.$$

The constant of proportionality in R depends on $\langle f(z(\ldots)\rangle$ factor in the denominator of (3.32). Regarding the effect of the non-uniform basic temperature gradient, we note that for any of the '$f(z)$ models' with similar boundaries, the only difference factor for the various $f(z)$'s is $\langle f(z(\ldots.)\rangle$ in the denominator of (3.30), (3.50), (3.55), (4.3), (4.8) and (4.17). The thermal profile effects are completely described by calculating these weightings, since they give R_{c_i}/R_{c_1} that are independent of Q and depend on the thermal depth ε. The influence of the various temperature profiles on R_{c_i}/R_{c_1} is shown in tables for each boundary combination. For all $f(z)$'s constructed such that $\langle f(z)\rangle = 1$, the thermal - profile effect is most effective for an unrealistic step function temperature jump concentrated at the maximum of WT function to augment convection. The step function results shown in the tables are merely values obtained from $\langle WT\,\delta(z-\varepsilon)\rangle$. From this we conclude that the step function temperature profile given by (3.45) augments convection, and the magnitude of augmentation will depend on the nature of the boundaries. Although augmentation is greatest for Free-Free adiabatic boundaries, we note that the case with a fixed isothermal lower boundary and a stress-free, adiabatic upper boundary with a step function temperature profile is a suitable mechanism for augmenting convection with magnetic field and an inverted parabolic temperature profile is an effective mechanism for suppressing convection compared to the other profiles.

Models 3 and 6 for edge cooling from above and Models 2 and 5 for heating from below, show the effects of anchoring the isothermal gradient near an edge, if WT is symmetric about the mid-line, it is then obvious that heating from below or cooling from above (i.e. symmetric $f(z)$) are equivalent. For example when both the boundaries are rigid and isothermal, WT given by (3.29) is symmetric under the exchange $z \leftrightarrow (1-z)$. Automatically $f(z)$ profiles that are mirror images of each other will give the same result. So $R_{c_2} = R_{c_3}$ and $R_{c_5} = R_{c_6}$ here. The same applies when both the boundaries are free and isothermal since $WT = z^2(1-z)^2[1 + z(1-z)]$ has the same symmetry. The same is also true in the case of rigid-rigid and free-free adiabatic boundaries.

If the symmetry is broken, in a practical case, where the lower boundary is rigid and the upper boundary is free, and they are either isothermal or adiabatic then cooling from above is more unstable than heating from below because from the tables we see that $R_{c_2} > R_{c_3}$ and $R_{c_5} > R_{c_6}$. Also, the greatest instability arises when this free upper boundary is adiabatic rather than isothermal. On the other hand, the greatest instability arises for the dynamically symmetric, free-free adiabatic boundaries.

To interpret the effects of a nonuniform temperature gradient on the onset of convection we consider the configuration in section 4.4. The thermal profiles effect is most effective ($\approx$ 48%) for an unrealistic, step function temperature jump concentrated at the maximum of *WT* function. For sudden heating from below, there is only a 6.3% profile effect which is far too small to see. For sudden cooling from above, which is representative of a possible situation occurring in nature and which can be achieved easily in the laboratory, the effects are more dramatic. Initially, the conduction state prevails (i.e. $R = 0$). For sudden cooling from above, the upper surface temperature suddenly reduced by an amount ΔT at $t = 0$ and remains at that temperature. As t increases, R initially decreases until it reaches some critical time at which it slowly starts to increase, eventually increasing at a faster rate and yielding the monotonically increasing curves. If the rate of cooling is sufficiently high, the R curve will intersect the R_c curve. Below this point, the system will be stable, beyond this point, it is unstable and magnetic field shifts the point of intersection. The Rayleigh number which is much lower than that for heating from below, existing at this intersecting point, is the critical Rayleigh number for this type and rate of cooling. A large amount of heat is required for conduction to establish the required value $R_{c_3} = 486.21$ for $Q = 0$ and $R_{c_3} = 1.0691 \times 10^5$ for $Q = 10^4$.

TABLE I

Critical Wave and Rayleigh Numbers for (a) Rigid-Rigid (b) Free-Free Isothermal Boundaries

Q	ac		$(R_{c_1} = R_{c_5} = R_{c_6}) \times 10^{-3}$		$(R_{c_2} = R_{c_3}) \times 10^{-3}$		$R_{c_4} \times 10^{-3}$	
	a	b	a	b	a	b	a	b
0	3.1165	2.2269	1.7499	0.6645	1.4014	0.5508	0.7999	0.3443
10^{-1}	3.1181	2.319	1.7525	0.6675	1.4034	0.5532	0.8011	0.3458
10^{0}	3.1321	2.2751	1.7751	0.6939	1.4215	0.5751	0.8115	0.3595
10^{1}	3.2599	2.5958	1.9967	0.9318	1.5990	0.7723	0.9122	0.4828
10^{2}	4.0068	3.7097	3.9500	2.6719	3.1632	2..2145	1.8057	1.3843
10^{3}	5.8954	5.6967	19.425	15.273	15.556	12.659	8.8801	7.9131
10^{4}	8.8971	8.6252	150.29	120.17	120.36	99.599	68.70	62.260
10^{5}	13.2731	12.8626	1354.2	1080.5	1084	895.62	619.10	559.85

TABLE II

Critical Rayleigh Nuymbers for (a) Rigid-Rigid (b) Free-Free Adiabatic Boundaries

Q	$(R_c)_1 \times 10^{-3}$		$(R_c)_2 = (R_c)_3) \times 10^{-3}$		$(R_c)_4 = \times 10^{-3}$	
	a	b	a	b	a	b
0	0.72	0.120	0.6011	0.1055	0.3840	0.0768
10^{-1}	0.7217	0.1212	0.6025	0.1066	0.3849	0.0776
10^{0}	0.7371	0.1321	0.6154	0.1162	0.3931	0.0946
10^{1}	0.8914	0.2414	0.7442	0.2124	0.7754	0.1545
10^{2}	2.4340	1.3342	2.0323	1.1736	1.2982	0.8939
10^{3}	17.860	12.262	14.913	10.786	9.5268	7.8482
10^{4}	172.12	121.52	143.72	106.91	91.812	77.791
10^{5}	1714.7	1214.4	1431.8	1068.2	914.66	777.21

From the Tables I to IV we observe that the critical Rayleigh numbers in the case of adiabatic boundaries are much smaller than those of isothermal boundaries, when $Q = 0$. This is because in the case of adiabatic boundaries, the surfaces are completely insulated and hence the energy produced will be stored in the region rather than dissipating to the surroundings. But in the presence of magnetic field, for large Q, the critical Rayleigh numbers in the case of adiabatic boundaries are larger than those of isothermal boundaries, because of the similar reason.

Comparing the values obtained from the present study with those of Chandrasekhar[1] and Nield[12] we found that the values are in good agreement with those of Chandrasekhar[1] when the boundaries are free-isothermal. The values deviate in other cases from those of Chandrasekhar[1] and Nield[12] for large values of Q. This may be due to the fact that the simple polynomial type of trial functions used in single term Galerkin expansion procedure does not take into account of Hartmann boundary layer that exists at the rigid boundaries for large Q. This has motivated us to determine the effect of Hartmann boundary layer on convection using asymptotic analysis.

In sections 5 and 6 therefore we have studied the effects of nonuniform basic temperature gradient and magnetic field on the onset of Rayleigh-Benard convection, using regular and singular perturbation techniques, with the idea of knowing how far the results obtained on using the single term Galerkin expansion procedure are true for the adiabatic boundaries, and also to estimate the error involved in the results obtained by the single term Galerkin expansion procedure. The single term Galerkin technique gives reasonable results only in the case of free-free boundaries, and deviates in the case of rigid-rigid boundaries for large values of Q. For example in the case of rigid-rigid adiabatic boundaries (4.14) yields the critical Rayleigh number

TABLE III

Critical wave and Rayleigh Numbers for

(a) Lower Rigid-Upper Free isothermal Boundaries

(b) Rigid-Fre Adiabatic Boundaries

(c) Lower Rigid Isothermal and Upper Free Adiabatic Boundaries

Q	a_c			$R_{c_1}\times10^{-3}$			$R_{c_2}\times10^{-3}$			$R_{c_3}\times10^{-3}$			$R_{c_4}\times10^{-3}$			$R_{c_5}\times10^{-3}$			$R_{c_6}\times10^{-3}$		
	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
0	2.6697	0	2.0521	1.1387	0.320	0.6912	0.9737	0.2926	0.6478	0.8762	0.2520	0.4862	0.5506	0.2846	0.3579	1.2336	0.3600	0.8732	1.0573	0.2880	0.5721
10^{-1}	2.6725	0	2.0543	1.1415	0.3215	0.6935	0.9762	0.294	0.6498	0.8784	0.2532	0.4877	0.5520	0.2855	0.3590	1.2367	0.3617	0.8760	1.0601	0.2894	0.5739
10^{0}	2.6968	0	2.0734	1.1672	0.3352	0.7135	0.9981	0.306	0.6686	0.8981	0.2640	0.5018	0.5644	0.1934	0.3694	1.2644	0.3771	0.9012	1.0838	0.3017	0.5905
10^{1}	2.9034	0	2.2361	1.4112	0.4720	0.9072	1.2067	0.4319	0.8501	1.0859	0.3719	0.6381	0.6823	0.2725	0.4696	1.5287	0.5314	1.1459	1.3104	0.4251	0.7508
10^{2}	3.8528	0	2.9881	3.955	1.8438	2.6139	2.9035	1.6860	2.4495	2.6128	1.4517	1.8385	1.6417	1.0638	1.3532	3.6785	2.0742	3.3018	3.1529	1.6594	2.1632
10^{3}	5.8273	0	4.5637	18.381	15.56	17.261	15.718	14.23	16.175	14.144	12.25	12.140	8.8875	8.9767	8.9362	19.91	17.502	2.1803	170.68	14.002	14.285
10^{4}	8.8194	0	6.9524	144.16	152.70	151.99	123.28	139.63	142.43	110.93	120.23	106.91	69.710	88.105	78.691	156.18	171.78	192.00	133.87	137.43	125.79
10^{5}	13.1601	0	10.4107	1299.7	1524.1	1450.4	1111.4	1393.6	1359.2	1000.2	1200.0	1020.2	628.45	887.94	750.92	1408.0	1714.6	1832.2	1206.8	1371.7	1200.4

TABLE IV

Critical Rayleigh Number for (a) Rigid-Rigid (b) Free-Free (c) Rigid-Free Adiabatic Boundaries (Using Regular Perturbation Technique)

Q	$(R_c)_1 \times 10^{-3}$			$(R_c)_2 \times 10^{-3}$			$(R_c)_3 \times 10^{-3}$			$(R_c)_4 \times 10^{-3}$			$(R_c)_5 \times 10^{-3}$			$(R_c)_6 \times 10^{-3}$		
	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
0	0.720	0.120	0.320	0.6001	0.1055	0.2926	0.6001	0.0768	0.2528	0.384	0.120	0.1846	0.720	0.120	0.360	0.720	0.720	0.288
10^{-1}	0.7217	0.1212	0.3215	0.6026	0.1066	0.294	0.6026	0.1066	0.2532	0.3853	0.0776	0.1868	0.7217	0.1212	0.3616	0.7217	0.1212	0.2894
10^{0}	0.7371	0.1321	0.3352	0.6156	0.1162	0.3064	0.6156	0.1162	0.2642	0.394	0.0847	0.1950	0.7371	0.1321	0.3767	0.7371	0.1321	0.302
10^{1}	0.8895	0.2413	0.4704	0.7447	0.2124	0.4291	0.7447	0.2124	0.3742	0.4802	0.1552	0.2772	0.8895	0.2413	0.5238	0.8895	0.2413	0.4269
10^{2}	2.3079	1.3274	1.73	1.9613	1.1738	1.57	1.9613	1.1738	1.44	1.3225	0.6895	1.06	2.308	1.3274	1.86	2.308	1.3274	1.63
10^{3}	14.4938	12.1364	13.2	12.721	10.777	11.9	12.719	10.7768	11.5	9.1636	8.0685	8.58	14.5938	12.1364	13.6	14.5938	12.1364	12.9
10^{4}	127.497	120.141	123.0	112.592	106.779	11.1	112.591	106.779	108.0	88.3376	80.1036	81.6	127.496	120.141	124.0	127.497	120.141	122.0
10^{5}	1222.30	1299.14	1210.0	1084.89	1066.78	1080.0	1084.89	1066.78	1070.0	810.654	800.459	805.0	1223.05	1200.14	1220.0	1223.05	1200.14	1200.0

$R_c = 720 + 17.14Q$, when $Q \to 0$ this gives $R_c \to 720$, the known value for ordinary viscous fluid. When $Q \to \infty$ this gives $R_c \to 17.14Q$. But for the case of rigid-rigid adiabatic boundaries the known value of R_c is $12Q$. Thus the value $R_c \to 17.14Q$ obtained using a single term Galerkin expansion is far too large compared to the known value $12Q$ obtained using the analytical method or regular perturbation technique. This inaccuracy for large values of Q may be attributed to the use of polynomial trial function in a single term Galerkin expansion which does not take care of the Hartmann boundary layer that inevitably arises for large values of Q at the rigid boundaries.

One of the best ways of dealing with the boundary layers, which arises for large Q is by the method of matched asymptotic expansion [10, 12, 24, 19]. Using this method, we found that the asymptotic value R_c is $12Q + 72Q^{1/2}$. We conclude that the asymptotic value $R_c \to 12Q$ is more accurate than the value $R_c \to \pi^2 Q$ obtained by Chandrasekhar[1] using the method of calculus of variation. The effect of Hartmann boundary layer which is of the order $Q^{1/2}$ is to increase the value of the critical Rayleigh number by an amount $72Q^{1/2}$. This is due to the fact that QC_0 does not vanish as $Q \to \infty$ in (6.5). The regular perturbation and the matched asymptotic techniques confirm a result obtained for the problem by the single term Galerkin method, viz, 'the asymptotic value of R crucially depends on the nonuniform basic temperature gradient $f(z)$'. In particular we conclude that the asymptotic value is the same for a particular $f(z)$ irrespective of the boundary combination. In other words the asymptotic value is invariant for all the boundary combinations and takes the value $12Q$, in contrast to the value $\pi^2 Q$ for isothermal boundaries with $f(z) = 1$ (see Chandrasekhar[1]).

ACKNOWLEDGEMENT

This work was supported by Department of Science and Technology, Government of India under Science and Engineering Research Council No. 111.5(36)/93-ET. I thank the referee for his valuable comments.

REFERENCES

1. S. Chandrasekhar, *Hydrodynamics and Hydromagnetic Stability*, Clarendon Press, Oxford (1961).
2. R. F. Danielson, *Theor. Astrophys. J.* **134** (1961), 289.
3. N. O. Weiss, *Proc. R. Soc.* **A 298** (1966), 310.
4. N. O. Weiss, *Proc. R. Soc. London* **A 361** (1978), 157.
5. N. O. Weiss, *J. Fluid Mech.* **108** (1981), 247.
6. F. H. Busse, *J. Fluid Mech.* **72** (1975), 67.
7. D. J. Galloway, M.R.E. Proctor and N. O. Weiss, *J. Fluid Mech.* **87** (1978), 243.
8. N. Rudraiah, *Astron. Soc. Japan* **33** (1981), 721.
9. N. Rudraiah, V. Kumudini and W. Unno, *Astron. Soc. Japan* **37** (1985), 183.
10. N. Rudraiah, *Effect of Nonuniform Temperature Gradient on Magnetoconvection* CANCAM, Canada.
11. N. Rudraiah, V. Ramachandramurthy and O. P. Chandna, *Int. J. Heat Mass Trans.* **28** (8) (1985), 1621-24.

12. D. A. Nield, Surface tension and byoyancy effects in the cellular convection of an electrically conducting liquid in a magnetic field *ZAMP* **17** (1966), 131.
13. N. Rudraiah, O. P. Chandna and M. R. Garg, *Indian J. Tech.* **24** (1986), 279.
14. N. Rudraiah and O. P. Chandna, *Can. J. Phys.* **64** (1) (1985), 90-99.
15. I. G. Currie, *J. Fluid Mech.* **29** (1967), 337.
16. B. A. Finlayson, *The Methods of Weighted Residual and Variational Principles*, Academic Press, 1972.
17. K. O. Mikaellian, *Phys. Rev.* **E 47** (1) (1993), 375-83.
18. E. M. Sparrow, R. J. Goldstein and V. K. Jonson, *J. Fluid Mech.* **18** (1964), 513.
19. N. Rudraiah and T. Masuoka, *Int. J. Engng. Sci.* **20** (1982), 27.
20. J. G. A. De Graff and VanderHeld, *Appl. Sci. Res. Sect.* **A 3** (1953), 393.
21. A. Vidal and A. Acrivos, *Indust. Engng. Chem. Fund.* **7** (1968), 53.
22. D. A. Nield, *J. Fluid Mech.* **29** (1967), 545.
23. D. A. Nield, *J. Fluid Mech.* **71** (1975), 441.
24. J. D. Cole, *Perturbation Methods in Applied Mathematics*, Blaisdell Publishing Co. (1968).